1. Differentialgleichungen

1.6 Lösen Sie die folgenden Differentialgleichungen mit der Methode der „Variation der Konstanten“.

1.6.1 $y' + \frac{1}{x^2} y = 3xe^{\frac{1}{x}} \qquad y(1) = e$

zug. hom.: $0 = y' + \frac{1}{x^2} y$

$$\frac{dy}{dx} = -\frac{y}{x^2}$$

$$\frac{dy}{y} = -\frac{dx}{x^2}$$

$$\int \frac{dy}{y} = -\int \frac{dx}{x^2}$$

$$\ln|y| = \frac{1}{x} + \tilde{C}$$

$$|y| = e^{\frac{1}{x}+\tilde{C}} = e^{\frac{1}{x}} e^{\tilde{C}}$$

$$y = \pm e^{\tilde{C}} e^{\frac{1}{x}}, \pm e^{\tilde{C}} = C$$

$$y = Ce^{\frac{1}{x}}$$

Variation der Konst.: $C: \quad C = C(x), \quad y = C(x)e^{\frac{1}{x}}$

$$y' = C'e^{\frac{1}{x}} + Ce^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)$$

in DGL:

$$C'e^{\frac{1}{x}} + \cancel{Ce^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)} + \cancel{\frac{1}{x^2}Ce^{\frac{1}{x}}} = 3xe^{\frac{1}{x}}$$

$$C'e^{\frac{1}{x}} = 3xe^{\frac{1}{x}}$$

$$C' = 3x \quad \left|\int(\ldots)\right.$$

$$C = \frac{3}{2}x^2 + K$$

$$\Rightarrow y = \left(\frac{3}{2}x^2 + K\right)e^{\frac{1}{x}} \quad \text{(allg. Lösung)}$$

$$y(1) = \left(\frac{3}{2} + K\right)e = e \quad \rightarrow K = -\frac{1}{2}$$

$$\Rightarrow y = \left(\frac{3}{2}x^2 - \frac{1}{2}\right)e^{\frac{1}{x}} \quad \text{(spez. Lösung)}$$

1.6.2 $y' + 2y = 5x^2 e^{-3x} \quad y(0) = 25$

zug. hom.: $y' + 2y = 0$

char. Gleichung: $\lambda + 2 = 0, \quad \lambda = -2$

$y_h = Ce^{-2x}$

Variation d. Konst.:

$$y = Ce^{-2x}$$

$$y' = C'e^{-2x} - 2Ce^{-2x}$$

in inhom. DGL:

$$C'e^{-2x} \cancel{-2Ce^{-2x}} \cancel{+2Ce^{-2x}} = 5x^2 e^{-3x}$$

$$C'e^{-2x} = 5x^2 e^{-3x} \quad \Big| : e^{-2x}$$

$$C' = 5x^2 e^{-x} \quad \Big\| \int (\ldots) dx$$

$$C = \int 5x^2 e^{-x} dx$$

mit part. Integration:

$$u = 5x^2, u' = 10x$$

$$v' = e^{-x}, v = -e^{-x}$$

$$C = -5x^2 e^{-x} + 10\int xe^{-x} dx$$

$$u = x, u' = 1$$

$$v' = e^{-x}, v = -e^{-x}$$

$$C = -5x^2 e^{-x} - 10xe^{-x} + 10\int e^{-x} dx$$

$$C = -5x^2 e^{-x} - 10xe^{-x} - 10e^{-x} + K$$

$$C = -e^{-x}\left(5x^2 + 10x + 10\right) + K$$

$$\Rightarrow y = \left[-e^{-x}\left(5x^2 + 10x + 10\right) + K\right] e^{-2x}$$

$$y = Ke^{-2x} - \left(5x^2 + 10x + 10\right) e^{-3x} \quad \text{(allg. Lösung)}$$

$$y(0) = 25$$

$$25 = K - 10 \;\Rightarrow\; K = 35$$

$$\Rightarrow y = 35e^{-2x} - \left(5x^2 + 10x + 10\right) e^{-3x} \text{ (spez. Lösung)}$$

1.6.3 $y' + 2y = \frac{e^{-2x}}{2x+1} \quad y(0) = 4$

$$\text{zug. hom.:} \quad y' + 2y = 0$$

$$\text{charakt. Gl.:} \quad \lambda + 2 = 0 \quad \rightarrow \quad \lambda = -2$$

$$y_h = Ce^{-2x}$$

Var. d. Konst.:

$$y = Ce^{-2x}$$

$$y' = C'e^{-2x} - 2Ce^{-2x}$$

in DGL:

$$C'e^{-2x} \cancel{-2Ce^{-2x}} \cancel{+2Ce^{-2x}} = \frac{e^{-2x}}{2x+1}$$

$$C'e^{-2x} = \frac{e^{-2x}}{2x+1} \quad \Big| : e^{-2x}$$

$$C' = \frac{1}{2x+1} \quad \Big| \int (\ldots)\,dx$$

$$C = \frac{1}{2}\ln|2x+1| + K$$

$$\Rightarrow y = \left(\frac{1}{2}\ln|2x+1| + K\right)e^{-2x} \quad (\text{allg. Lösung})$$

$$y(0) = \ln(1) + K = 4$$

$$K = 4$$

$$\Rightarrow y = \left(\frac{1}{2}\ln|2x+1| + 4\right)e^{-2x} \quad (\text{spez. Lösung})$$

1.6.4 $y' + 5y = \dfrac{e^{-5x}}{4x^2+1} \qquad y(0) = 0$

$$\text{zug. hom.:} \quad y' + 5y = 0$$

$$\text{charakt. Gl.:} \quad \lambda + 5 = 0 \quad \rightarrow \quad \lambda = -5$$

$$y_h = Ce^{-5x}$$

Var. d. Konst.:

$$y = Ce^{-5x}$$

$$y' = C'e^{-5x} - 5Ce^{-2x}$$

in DGL:

$$C'e^{-5x} \cancel{-5Ce^{-5x}} \cancel{+5Ce^{-5x}} = \frac{e^{-5x}}{4x^2+1}$$

$$C'e^{-5x} = \frac{e^{-5x}}{4x^2+1} \quad \Big| : e^{-5x}$$

$$C' = \frac{1}{4x^2+1} \quad \Big| \int (\ldots)\,dx$$

$$C = \int \frac{dx}{4x^2+1} = \frac{1}{2}\arctan(u) + K$$

lin. Substitution: $\quad 2x = u, \ \dfrac{du}{dx} = 2, \ dx = \dfrac{1}{2}du$

$$C = \frac{1}{2}\arctan(2x) + K$$

$$\Rightarrow y = \left[\frac{1}{2}\arctan(2x) + K\right]e^{-5x} \quad \text{(allg. Lösung)}$$

$$y(0) = K = 0$$

$$\Rightarrow y = \frac{1}{2}\arctan(2x)e^{-5x} \quad \text{(spez. Lösung)}$$

1.6.5 $\sin(x)y + \cos(x)y' = x\cos^3(x) \quad y(0) = -2$

zug. hom.: $\sin(x)y + \cos(x)y' = 0$

$$\frac{dy}{dx}\cos(x) = -\sin(x)y$$

$$\frac{dy}{y} = -\frac{\sin(x)}{\cos(x)}dx \quad \Big|\int(\ldots)$$

$$\ln|y| = \ln|\cos(x)| + \tilde{C}$$

$$\ln|y| - \ln|\cos(x)| = \tilde{C}$$

$$\ln\left|\frac{y}{\cos(x)}\right| = \tilde{C}$$

$$\left|\frac{y}{\cos(x)}\right| = e^{\tilde{C}}, \text{mit } C = \pm e^{\tilde{C}}$$

$$\frac{y}{\cos(x)} = C$$

$$y_h = C\cos(x)$$

Var. d. Konst.: C ist nun Funktion von x

$$y = C\cos(x)$$

$$y' = C'\cos(x) - C\sin(x)$$

in DGL:

$$x\cos^3(x) = \sin(x)\big(C\cos(x)\big) + \cos(x)\big(C'\cos(x) - C\sin(x)\big)$$

$$x\cos^3(x) = \cancel{C\cos(x)\sin(x)} + C'\cos^2(x) \cancel{-C\cos(x)\sin(x)}$$

$$x\cos^3(x) = C'\cos^2(x)$$

$$C' = x\cos(x) \quad \Big|\int(\ldots)dx$$

$$C = \int x\cos(x)dx = x\sin(x) - \int \sin(x)dx$$

$$u = x, u' = 1$$

$$v' = \cos(x), v = \sin(x)$$

$$C = x\sin(x) + \cos(x) + K$$

$$\Rightarrow y = \big(x\sin(x) + \cos(x) + K\big)\cos(x) \quad \text{(allg. Lösung)}$$

$$y(0) = 1 + K = -2$$

$$K = -3$$

$$\Rightarrow y = \big(x\sin(x) + \cos(x) - 3\big)\cos(x) \quad \text{(spez. Lösung)}$$

1.6.6 $y' + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{x^2 + 2x + 1} \quad y(0) = 2$

zug. hom.: $y' + \frac{y}{\sqrt{x}} = 0$

$$\frac{dy}{dx} = -\frac{y}{\sqrt{x}}$$

$$\frac{dy}{y} = -\frac{dx}{\sqrt{x}} \quad \Big| \int (\ldots)$$

$$\ln|y| = -2\sqrt{x} + \tilde{C}$$

$$|y| = e^{-2\sqrt{x} + \tilde{C}} = e^{-2\sqrt{x}} e^{\tilde{C}}, \quad C = \pm e^{\tilde{C}}$$

$$y = Ce^{-2\sqrt{x}}$$

Var. d. Konst.: C ist nun eine Funktion von x

$$y = Ce^{-2\sqrt{x}}$$

$$y' = C'e^{-2\sqrt{x}} + Ce^{-2\sqrt{x}} \left(-2 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} \right)$$

$$y' = C'e^{-2\sqrt{x}} - \frac{C}{\sqrt{x}} e^{-2\sqrt{x}}$$

in DGL:

$$C'e^{-2\sqrt{x}} - \cancel{\frac{Ce^{-2\sqrt{x}}}{\sqrt{x}}} + \cancel{\frac{Ce^{-2\sqrt{x}}}{\sqrt{x}}} = \frac{e^{-2\sqrt{x}}}{x^2 + 2x + 1}$$

$$C' = \frac{1}{x^2 + 2x + 1}$$

$$C' = \frac{1}{(x+1)^2} \quad \Big| \int (\ldots) dx$$

$$C = -\frac{1}{x+1} + K$$

$$\Rightarrow y = \left(-\frac{1}{x+1} + K \right) e^{-2\sqrt{x}} \quad \text{(allg. Lösung)}$$

$$y(0) = K - 1 = 2 \to K = 3$$

$$\Rightarrow y = \left(3 - \frac{1}{x+1} \right) e^{-2\sqrt{x}}$$

$$\Rightarrow y = \left(\frac{3x+2}{x+1} \right) e^{-2\sqrt{x}} \quad \text{(spez. Lösung)}$$