2. Laplace-Transformation

Aufgabe 2.1 Nachfolgend für $t \geq 0$ definierte Funktionen haben für $t < 0$ den Funktionswert $f(t) = 0$. Transformieren Sie mit der Tabelle in den Bildbereich der Laplace-Transformation!

2.1.1. $f(t) = 4e^{-3t}$

$$f(t) = 4e^{-3t} \multimap F(s) = \frac{4}{s+3}$$

2.1.2. $f(t) = 1 + 2t + t^2$

$$f(t) = 1 + 2t + t^2 \multimap F(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}$$

2.1.3. $f(t) = (1 + 2t + t^2)e^{-2t}$

$$f(t) = (1 + 2t + t^2)e^{-2t} \multimap F(s) = \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{2}{(s+2)^3}$$

2.1.4. $f(t) = e^{t} + e^{-t}$

$$f(t) = e^{t} + e^{-t} \qquad (= 2\cosh(t))$$

$$\multimap F(s) = \frac{1}{s-1} + \frac{1}{s+1}$$

$$\multimap F(s) = \frac{2s}{s^2 - 1}$$

2.1.5. $f(t) = \frac{1}{2}\left(e^{t} - e^{-t}\right)$

$$f(t) = \frac{1}{2}\left(e^{t} - e^{-t}\right) \qquad \left(= \sinh(t)\right)$$

$$\multimap F(s) = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

$$\multimap F(s) = \frac{1}{2}\left(\frac{2}{s^2-1}\right)$$

$$\multimap F(s) = \frac{1}{s^2-1}$$

2.1.6. $f(t) = \sin(2t) + 3\cos(2t)$

$$f(t) = \sin(2t) + 3\cos(2t)$$

$$\multimap F(s) = \frac{2}{s^2+4} + \frac{3s}{s^2+4} = \frac{2+3s}{s^2+4}$$

2.1.7. $f(t) = e^{-4t}\left[\sin(2t) + 3\cos(2t)\right]$

$$f(t) = e^{-4t}\left[\sin(2t) + 3\cos(2t)\right]$$

$$\multimap F(s) = \frac{2}{(s+4)^2+4} + \frac{3(s+4)}{(s+4)^2+4} = \frac{3s+14}{s^2+8s+20}$$

Aufgabe 2.2 Berechnen Sie die Bildfunktion unter Verwendung des Integrals, welches die Laplace-Transformation definiert.

2.2.1. $$f(t) = \begin{cases} 0 & t < 0 \\ 2 & 0 \le t \le 2\pi \\ 2\cos(3t) & 2\pi < t \end{cases}$$

$$F(s)=\int_0^{\infty} f(t)e^{-st}dt$$

$$=\lim_{b\to\infty}\int_0^{2\pi} 2e^{-st}dt+\int_{2\pi}^{b} 2\cos(3t)e^{-st}dt$$

$$=\lim_{b\to\infty}\left\{\left[-\frac{2}{s}e^{-st}\right]_0^{2\pi}+\left[\frac{2}{s^2+9}e^{-st}\left(3\sin(3t)-s\cos(3t)\right)\right]_{2\pi}^{b}\right\}$$

$$=-\frac{2}{s}e^{-2\pi s}-\frac{2}{s}e^{0}$$

$$+\lim_{b\to\infty}\left\{\frac{2}{s^2+9}\left(3\sin(3b)\underbrace{e^{-sb}}_{\to 0}-s\cos(3b)\underbrace{e^{-sb}}_{\to 0}-3\underbrace{\sin(3\cdot 2\pi)}_{=0}e^{-2\pi s}-s\underbrace{\cos(3\cdot 2\pi)}_{=1}e^{-2\pi s}\right)\right\}$$

$$=-\frac{2}{s}e^{-2\pi s}-\frac{2}{s}-\frac{2se^{-2\pi s}}{s^2+9}$$

$$=-\frac{2}{s}\left(e^{-2\pi s}+1\right)-\frac{2s}{s^2+9}e^{-2\pi s}$$

2.2.2.

$$f(t)=\begin{cases}0 & t<0\\ A & 0\le t\le 3\\ Ae^{-(t-3)} & 3<t\end{cases}$$

$$F(s)=\int_0^{\infty} f(t)e^{-st}dt$$

$$=\lim_{b\to\infty}\int_0^{3} Adt+\int_3^{b} Ae^{-(t-3)}e^{-st}dt=\lim_{b\to\infty}\int_0^{3} Adt+\int_3^{b} Ae^{-t(s+1)+3}dt$$

$$=\lim_{b\to\infty}\left\{\left[-\frac{A}{s}e^{-st}\right]_0^{3}+\left[-\frac{A}{s+1}e^{-t(s+1)+3}\right]_3^{b}\right\}$$

$$=\lim_{b\to\infty}\left\{-\frac{A}{s}e^{-3s}+\frac{A}{s}e^{0}-\frac{A}{s+1}e^{-b(s+1)+3}+\frac{A}{s+1}e^{-3(s+1)+3}\right\}$$

$$=\lim_{b\to\infty}\left\{-\frac{A}{s}e^{-3s}+\frac{A}{s}-\frac{A}{s+1}\underbrace{e^{-b(s+1)+3}}_{\to 0}+\frac{A}{s+1}e^{-3s}\right\}$$

$$=-\frac{A}{s}e^{-3s}+\frac{A}{s}+\frac{A}{s+1}e^{-3s}$$

Aufgabe 2.3 Transformieren Sie mit der Tabelle der Laplace-Korrespondenzen vom Bild- in den Zeitbereich. Falls nötig, zerlegen Sie die Ausdrücke zunächst in Partialbrüche.

2.3.1. $F(s) = \frac{1}{s+2}$ $\quad \left| L^{-1}\{\ldots\} \right. \quad \Rightarrow f(t) = e^{-2t}$

2.3.2. $F(s) = \frac{5}{s+2}$ $\quad \left| L^{-1}\{\ldots\} \right. \quad \Rightarrow f(t) = 5e^{-2t}$

2.3.3. $F(s) = \frac{2}{s^2-4}$

$$F(s) = \frac{2}{s^2-4}$$
$$\frac{2}{s^2-4} = \frac{A}{s+2} + \frac{B}{s-2}$$
$$\frac{2}{s^2-4} = \frac{A(s-2)+B(s+2)}{(s+2)(s-2)}$$

Zählervergleich:

$$2 = A(s-2) + B(s+2)$$

1.Weg – Nullstellen einsetzen:

$$s = 2 \to 2 = 4B$$
$$s = -2 \to 2 = -4A$$

2.Weg – Koeffizientenvergleich:

$$s^1 : 0 = A + B$$
$$s^0 : 2 = -2A + 2B$$

$$A = -\frac{1}{2}; \quad B = \frac{1}{2}$$

$$F(s) = -\frac{1}{2}\frac{1}{s+2} + \frac{1}{2}\frac{1}{s-2} \quad \left| L^{-1}\{\ldots\} \right.$$
$$f(t) = \frac{1}{2}\left(e^{2t} - e^{-2t}\right)$$
$$f(t) = \sinh(2t)$$

2.3.4. $F(s) = \frac{2}{s^2+4} \quad \left| L^{-1}\{\ldots\} \right. \quad \Rightarrow f(t) = \sin(2t)$

2.3.5. $F(s) = \frac{s}{s^2+4} \quad \left| L^{-1}\{\ldots\} \right. \quad \Rightarrow f(t) = \cos(2t)$

2.3.6. $F(s) = \frac{s}{s^2-4}$

$$F(s) = \frac{s}{s^2-4}$$

$$\frac{s}{s^2-4} = \frac{A}{s+2} + \frac{B}{s-2}$$

$$\frac{s}{s^2-4} = \frac{A(s-2)+B(s+2)}{(s+2)(s-2)}$$

Zählervergleich:

$$s = A(s-2)+B(s+2)$$

1.Weg – Nullstellen einsetzen:

$$s = 2 \to 2 = 4B$$

$$s = -2 \to -2 = -4B$$

2.Weg – Koeffizientenvergleich:

$$s^1: 1 = A + B$$

$$s^0: 0 = -2A + 2B$$

$$A = \frac{1}{2}; \quad B = \frac{1}{2}$$

$$F(s) = \frac{1}{2}\frac{1}{s+2} + \frac{1}{2}\frac{1}{s-2} \quad \left| L^{-1}\{\ldots\} \right.$$

$$f(t) = \frac{1}{2}\left(e^{2t} + e^{-2t}\right)$$

$$f(t) = \cosh(2t)$$

2.3.7. $F(s) = \frac{1}{(s+2)^2}$

$F(s) = \frac{1}{(s+2)^2} \quad \left| L^{-1}\{\ldots\} \right.$ Dämpfungsregel

$f(t) = te^{-2t}$

2.3.8. $F(s) = \frac{s}{(s+2)^2}$

1.Weg – Partialbruchzerlegung :

$$F(s) = \frac{s}{(s+2)^2} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} = \frac{A(s+2)+B}{(s+2)^2}$$

Zählervergleich :

$$s = A(s+2)+B$$

$$s = -2 \;\rightarrow\; \quad -2 = B$$

$$s^1 : \quad \rightarrow \quad 1 = A$$

$$F(s) = \frac{1}{(s+2)} + \frac{-2}{(s+2)^2} \qquad \Big| L^{-1}$$

$$f(t) = e^{-2t} - 2te^{-2t}$$

2.Weg – Differentationsregel :

$$F(s) = s\frac{1}{(s+2)^2} = s \cdot G(s)$$

$$G(s) = \frac{1}{(s+2)^2} \qquad \text{zu} \qquad g(t) = te^{-2t}$$

$$f(t) = \frac{d}{dt}g(t) = \frac{d}{dt}\left(te^{-2t}\right)$$

$$f(t) = e^{-2t} - 2te^{-2t}$$

2.3.9. $F(s) = \frac{1}{(s+5)^3}$

$$F(s) = \frac{1}{(s+5)^3} \qquad \Big| L^{-1}\{\ldots\} \qquad \text{Dämpfungsregel}$$

$$f(t) = \frac{t^2}{2}e^{-5t}$$

2.3.10. $F(s)=\frac{s}{(s+5)^3}$

1.Weg – Partialbruchzerlegung :

$$F(s)=\frac{s}{(s+5)^3}=\frac{A}{(s+5)}+\frac{B}{(s+5)^2}+\frac{C}{(s+5)^3}$$

$$F(s)=\frac{A(s+5)^2+B(s+5)+C}{(s+5)^3}$$

Zählervergleich :

$$s=A(s+5)^2+B(s+5)+C$$

$$s=-5 \quad \rightarrow \quad -5=C$$

$$s^2: \quad \rightarrow \quad 0=A$$

$$s^1: \quad \rightarrow \quad 1=10A+B \quad \rightarrow \quad B=1$$

$$F(s)=\frac{1}{(s+5)^2}+\frac{-5}{(s+5)^3} \qquad \Big| L^{-1}$$

$$f(t)=te^{-5t}-\frac{5}{2}t^2e^{-5t}$$

2.Weg – Differentationsregel :

$$F(s)=s\frac{1}{(s+5)^3}=s\cdot G(s)$$

$$G(s)=\frac{1}{(s+5)^3} \qquad \text{mit} \qquad g(t)=\frac{t^2}{2}e^{-5t}$$

$$f(t)=\frac{d}{dt}g(t)=\frac{d}{dt}\left(\frac{t^2}{2}e^{-5t}\right)$$

$$f(t)=\left(t-\frac{5}{2}t^2\right)e^{-5t}$$

2.3.11. $F(s)=\frac{s^2}{(s+5)^3}$

1.Weg – Partialbruchzerlegung :

$$F(s)=\frac{s^2}{(s+5)^3}=\frac{A}{(s+5)}+\frac{B}{(s+5)^2}+\frac{C}{(s+5)^3}$$

$$F(s)=\frac{A(s+5)^2+B(s+5)+C}{(s+5)^3}$$

Zählervergleich :

$$s^2=A(s+5)^2+B(s+5)+C$$

$$s=-5 \rightarrow \quad 25=C$$

$$s^2: \quad \rightarrow \quad 1=A$$

$$s^1: \quad \rightarrow \quad 0=10A+B \quad \rightarrow \quad B=-10$$

$$F(s)=\frac{1}{s+5}+\frac{-10}{(s+5)^2}+\frac{25}{(s+5)^3} \qquad \Big| L^{-1}$$

$$f(t)=e^{-5t}-10te^{-5t}-\frac{25}{2}t^2e^{-5t}$$

2.Weg – Differentationsregel :

$$F(s)=s^2\frac{1}{(s+5)^3}=s^2\cdot G(s)$$

$$G(s)=\frac{1}{(s+5)^3} \qquad \text{mit} \qquad g(t)=\frac{t^2}{2}e^{-5t}$$

$$f(t)=\frac{d^2}{dt^2}g(t)=\frac{d^2}{dt^2}\left(\frac{t^2}{2}e^{-5t}\right)$$

$$f(t)=\left(\frac{25}{2}t^2-10t+1\right)e^{-5t}$$

2.3.12. $F(s) = \frac{1}{s^2 + 6s + 13}$

$s^2 + 6s + 13 = 0 \quad \Rightarrow \quad s_{1/2} = -3 \pm 2j$

Der Nenner hat konjugiert komplexe Nullstellen.
Deshalb ist quadratische Ergänzung zweckmäßiger als Partialbruchzerlegung

$$F(s) = \frac{1}{(s+3)^2 + 4} = \frac{1}{2} \cdot \frac{2}{(s+3)^2 + 2^2} \qquad \Big| L^{-1}$$

$$f(t) = \frac{1}{2}\sin(2t)e^{-3t}$$

2.3.13. $F(s) = \frac{s}{s^2 + 6s + 13}$

Wie in 3.12 wird auch hier die qadratische Ergänzung benutzt

$$F(s) = \frac{s}{(s+3)^2 + 4} \qquad \text{gedämpfter Kosinus erfordert überall } s+3$$

$$F(s) = \frac{(s+3) - 3}{(s+3)^2 + 4}$$

$$F(s) = \frac{(s+3) - 3}{(s+3)^2 + 4} = \frac{(s+3)}{(s+3)^2 + 4} - \frac{3}{(s+3)^2 + 4}$$

$$F(s) = \frac{(s+3)}{(s+3)^2 + 2^2} - \frac{3}{2} \cdot \frac{2}{(s+3)^2 + 2^2} \qquad \Big| L^{-1}$$

$$f(t) = e^{-3t}\left[\cos(2t) - \frac{3}{2}\sin(2t)\right]$$

2.3.14. $F(s) = \frac{s+1}{s^2+6s+13}$

Wie in 3.12 wird auch hier die qadratische Ergänzung benutzt

$$F(s) = \frac{s+1}{s^2+6s+13}$$
$$F(s) = \frac{(s+3)-2}{(s+3)^2+4}$$
$$F(s) = \frac{(s+3)}{(s+3)^2+2^2} - \frac{2}{(s+3)^2+2^2} \qquad \left| L^{-1}\{\ldots\}\right.$$

$$f(t) = e^{-3t}\left[\cos(2t) - \sin(2t)\right]$$

Aufgabe 2.4 Führen Sie die Partialbruchzerlegung durch und transformieren Sie danach in den Zeitbereich.

2.4.1. $F(s) = \frac{1}{s(s+1)}$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$
$$\frac{1}{s(s+1)} = \frac{A(s+1)+Bs}{s(s+1)}$$

Zählervergleich :

$$1 = A(s+1) + Bs$$

Nullstellen einsetzen:

$$s = 0 \Rightarrow 1 = A$$
$$s = -1 \Rightarrow 1 = -B, \quad B = -1$$

$$F(s) = \frac{1}{s} - \frac{1}{s+1} \qquad \left| L^{-1}\{\ldots\}\right.$$
$$f(t) = 1 - e^{-t}$$

2.4.2. $F(s) = \frac{s+2}{(s-2)(s+3)}$

$$\frac{s+2}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$$\frac{s+2}{(s-2)(s+3)} = \frac{A(s+3)+B(s-2)}{(s-2)(s+3)}$$

$$s+2 = A(s+3)+B(s-2)$$

Nullstellen einsetzen:

$$s = 2 \Rightarrow 4 = 5A, \quad A = \frac{4}{5}$$

$$s = -3 \Rightarrow -1 = -5B, \quad B = \frac{1}{5}$$

$$F(s) = \frac{4}{5}\frac{1}{s-2} + \frac{1}{5}\frac{1}{s+3} \quad \left| L^{-1}\{\ldots\}\right.$$

$$f(t) = \frac{4}{5}e^{2t} + \frac{1}{5}e^{-3t}$$

2.4.3. $F(s) = \frac{s}{(s+1)(s+2)^2}$

$$\frac{s}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$\frac{s}{(s+1)(s+2)^2} = \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2}$$

$$\frac{s}{(s+1)(s+2)^2} = \frac{A(s^2+4s+4) + B(s^2+3s+2) + C(s+1)}{(s+1)(s+2)^2}$$

$$s = A(s^2+4s+4) + B(s^2+3s+2) + C(s+1)$$

Nullstellen einsetzen:

$$s = -1 \Rightarrow -1 = A$$

$$s = -2 \Rightarrow -2 = -C$$

Koeff.-Vergleich:

$$s^2 : 0 = A + B$$

$$A = -1; \quad B = 1; \quad C = 2$$

$$F(s) = -\frac{1}{s+1} + \frac{1}{s+2} + \frac{2}{(s+2)^2} \quad \left| L^{-1}\{\ldots\}\right.$$

$$f(t) = -e^{-t} + e^{-2t} + 2te^{-2t}$$

2.4.4. $F(s) = \frac{s}{(s+1)(s^2+2)}$

$$\frac{s}{(s+1)(s^2+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$$

$$\frac{s}{(s+1)(s^2+2)} = \frac{A(s^2+2)+(Bs+C)(s+1)}{(s+1)(s^2+2)}$$

Zählervergleich:

$$s = A(s^2+2)+(Bs+C)(s+1)$$

Nullstellen einsetzen:

$$s = -1 \qquad \Rightarrow \qquad -1 = 3A$$

Koeff.-Vergleich:

$$s^2 : 0 = A + B$$

$$s^0 : 0 = 2A + C$$

$$A = \frac{-1}{3}; \quad B = \frac{1}{3}; \quad C = \frac{2}{3}$$

$$F(s) = \frac{-1}{3}\frac{1}{s+1} + \frac{1}{3}\frac{s+2}{s^2+2}$$

$$F(s) = \frac{-1}{3}\frac{1}{s+1} + \frac{1}{3}\frac{s}{s^2+2} + \frac{\sqrt{2}}{3}\frac{\sqrt{2}}{s^2+2}$$

$$f(t) = \frac{-1}{3}e^{-t} + \frac{1}{3}\cos\left(\sqrt{2}t\right) + \frac{\sqrt{2}}{3}\sin\left(\sqrt{2}t\right)$$

2.4.5. $F(s) = \frac{2s+3}{(s-1)(s^2+4s+29)}$

$$s^2 + 4s + 29 = 0 \qquad \Rightarrow \qquad s_{1/2} = -2 \pm 5j$$

$\Rightarrow$ qudratischen Term zusammen lassen

$$\frac{2s+3}{(s-1)(s^2+4s+29)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4s+29}$$

$$\frac{2s+3}{(s-1)(s^2+4s+29)} = \frac{A(s^2+4s+29)+(Bs+C)(s-1)}{(s-1)(s^2+4s+29)}$$

$$2s+3 = A(s^2+4s+29) + B(s^2-s) + C(s-1)$$

Nullstellen einsetzen:

$$s = 1 \quad \Rightarrow \quad 5 = 34A$$

Koeff.-Vergleich:

$$s^2 : 0 = A + B$$

$$s^0 : 3 = 29A - C$$

$$A = \frac{5}{34}; \quad B = -\frac{5}{34}; \quad C = \frac{43}{34}$$

$$F(s) = \frac{5}{34}\frac{1}{s-1} - \frac{5}{34}\frac{s}{s^2+4s+29} + \frac{43}{34}\frac{1}{s^2+4s+29}$$

$$F(s) = \frac{5}{34}\frac{1}{s-1} - \frac{5}{34}\frac{(s+2)}{(s+2)^2+25} + \frac{53}{34}\cdot\frac{1}{5}\cdot\frac{5}{(s+2)^2+25} \qquad \left| L^{-1}\{\ldots\}\right.$$

$$f(t) = \frac{5}{34}e^t - \frac{5}{34}\cos(5t)e^{-2t} + \frac{53}{170}\sin(5t)e^{-2t}$$

$$f(t) = \frac{5}{34}e^t - \frac{e^{-2t}}{170}\left[25\cos(5t) - 53\sin(5t)e^{-2t}\right]$$

Aufgabe 2.5 Lösen Sie folgende Anfangswertaufgaben mit der Laplace-Transformation.

2.5.1. $\ddot{x} - 4\dot{x} - 5x = 0$; $x(0) = 1, \dot{x}(0) = -2$

$$\ddot{x} - 4\dot{x} - 5x = 0 \quad \Big| L\{\ldots\}$$
$$0 = s^2X(s) - sx(0) - \dot{x}(0) - 4sX(s) + 4x(0) - 5X(s)$$
$$0 = s^2X(s) - s + 2 - 4sX(s) + 4 - 5X(s)$$
$$s - 6 = X(s)(s^2 - 4s - 5)$$
$$X(s) = \frac{s-6}{s^2 - 4s - 5}$$
$$\frac{s-6}{s^2 - 4s - 5} = \frac{A}{(s+1)} + \frac{B}{(s-5)}$$
$$\frac{s-6}{s^2 - 4s - 5} = \frac{A(s-5) + B(s+1)}{s^2 - 4s - 5}$$
$$s - 6 = A(s-5) + B(s+1)$$

Nullstellen einsetzen:

$$s = 5 \Rightarrow 1 = 6B$$
$$s = -1 \Rightarrow -7 = -6A$$
$$A = \frac{7}{6}; \quad B = -\frac{1}{6}$$
$$X(s) = \frac{7}{6}\frac{1}{(s+1)} - \frac{1}{6}\frac{1}{(s-5)} \quad \Big| L^{-1}\{\ldots\}$$
$$x(t) = \frac{7}{6}e^{-t} - \frac{1}{6}e^{5t}$$

2.5.2. $\ddot{x} - 4\dot{x} - 5x = 0$; $x(0) = 0, \dot{x}(0) = 0$

$$\ddot{x} - 4\dot{x} - 5x = 0 \quad \Big| L\{\ldots\}$$
$$0 = s^2X(s) - sx(0) - \dot{x}(0) - 4sX(s) + 4x(0) - 5X(s)$$
$$0 = s^2X(s) - 4sX(s) - 5X(s)$$
$$0 = X(s)(s^2 - 4s - 5)$$
$$X(s) = \frac{0}{s^2 - 4s - 5}$$
$$X(s) = 0 \quad \Big| L^{-1}\{\ldots\}$$
$$x(t) \equiv 0 = \text{const.}$$

2.5.3. $\ddot{x} - 4\dot{x} - 5x = 3e^{2t}$; $\quad x(0) = 0,\ \dot{x}(0) = 0$

$$\ddot{x} - 4\dot{x} - 5x = 3e^{2t} \quad \left| L\{\ldots\}\right.$$

$$\frac{3}{s-2} = s^2X(s) - sx(0) - \dot{x}(0) - 4sX(s) + 4x(0) - 5X(s)$$

$$\frac{3}{s-2} = s^2X(s) - 4sX(s) - 5X(s)$$

$$\frac{3}{s-2} = X(s)\left(s^2 - 4s - 5\right)$$

$$X(s) = \frac{3}{(s-2)\left(s^2 - 4s - 5\right)}$$

$$\frac{3}{(s-2)\left(s^2 - 4s - 5\right)} = \frac{A}{(s-2)} + \frac{B}{(s+1)} + \frac{C}{(s-5)}$$

$$\frac{3}{(s-2)\left(s^2 - 4s - 5\right)} = \frac{A(s+1)(s-5) + B(s-2)(s-5) + C(s-2)(s+1)}{(s-2)(s+1)(s-5)}$$

Zählervergleich:

$$3 = A(s+1)(s-5) + B(s-2)(s-5) + C(s-2)(s+1)$$

Nullstellen einsetzen:

$$s = 2 \quad \Rightarrow 3 = -9A$$

$$s = 5 \quad \Rightarrow 3 = 18C$$

$$s = -1 \quad \Rightarrow 3 = 18B$$

$$A = -\frac{1}{3}; \quad B = \frac{1}{6}; \quad C = \frac{1}{6}$$

$$X(s) = -\frac{1}{3}\frac{1}{(s-2)} + \frac{1}{6}\frac{1}{(s+1)} + \frac{1}{6}\frac{1}{(s-5)} \quad \left| L^{-1}\{\ldots\}\right.$$

$$x(t) = -\frac{1}{3}e^{2t} + \frac{1}{6}e^{-t} + \frac{1}{6}e^{5t}$$

2.5.4. $\ddot{x} - 4\dot{x} - 5x = \sin(3t)$; $x(0) = 0,\ \dot{x}(0) = 0$

$$\ddot{x} - 4\dot{x} - 5x = \sin(3t) \quad \Big| L\{\ldots\}$$

$$\frac{3}{s^2+9} = s^2X(s) - sx(0) - \dot{x}(0) - 4sX(s) + 4x(0) - 5X(s)$$

$$\frac{3}{s^2+9} = s^2X(s) - 4sX(s) - 5X(s)$$

$$\frac{3}{s^2+9} = X(s)\left(s^2 - 4s - 5\right)$$

$$X(s) = \frac{3}{\left(s^2+9\right)\left(s^2-4s-5\right)}$$

$$\frac{3}{\left(s^2+9\right)\left(s^2-4s-5\right)} = \frac{As+B}{\left(s^2+9\right)} + \frac{C}{(s+1)} + \frac{D}{(s-5)}$$

$$\frac{3}{\left(s^2+9\right)\left(s^2-4s-5\right)} = \frac{(As+B)(s+1)(s-5) + C\left(s^2+9\right)(s-5) + D\left(s^2+9\right)(s+1)}{\left(s^2+9\right)(s+1)(s-5)}$$

$$3 = (As+B)(s+1)(s-5) + C\left(s^2+9\right)(s-5) + D\left(s^2+9\right)(s+1)$$

Nullstellen einsetzen:

$$s = 5 \quad \Rightarrow 3 = 204D$$

$$s = -1 \quad \Rightarrow 3 = -60C$$

Koeff.-Vergleich:

$$s^3 : 0 = A + C + D$$

$$s^0 : 3 = -5B - 45C + 9D$$

$$A = \frac{3}{85};\quad B = \frac{81}{170};\quad C = -\frac{1}{20};\quad D = \frac{1}{68}$$

$$X(s) = \frac{As+B}{\left(s^2+9\right)} + \frac{C}{(s+1)} + \frac{D}{(s-5)}$$

$$X(s) = \frac{\frac{3}{85}s + \frac{81}{170}}{s^2+9} - \frac{1}{20}\frac{1}{(s+1)} + \frac{1}{68}\frac{1}{(s-5)}$$

$$X(s) = \frac{3}{85}\frac{s}{s^2+9} + \frac{27}{170}\frac{3}{s^2+9} - \frac{1}{20}\frac{1}{(s+1)} + \frac{1}{68}\frac{1}{(s-5)} \quad \Big| L^{-1}\{\ldots\}$$

$$x(t) = \frac{3}{85}\cos(3t) + \frac{27}{170}\sin(3t) - \frac{1}{20}e^{-t} - \frac{1}{68}e^{5t}$$

2.5.5. $\ddot{x} - 4\dot{x} - 5x = \cos(4t)$; $x(0) = 0,\ \dot{x}(0) = 0$

$$\ddot{x} - 4\dot{x} - 5x = \cos(4t) \quad \Big| \mathcal{L}\{\ldots\}$$

$$\frac{s}{s^2+16} = s^2X(s) - sx(0) - \dot{x}(0) - 4sX(s) + 4x(0) - 5X(s)$$

$$\frac{s}{s^2+16} = s^2X(s) - 4sX(s) - 5X(s)$$

$$\frac{s}{s^2+16} = X(s)\left(s^2 - 4s - 5\right)$$

$$X(s) = \frac{s}{\left(s^2+16\right)\left(s^2-4s-5\right)}$$

$$\frac{s}{\left(s^2+16\right)\left(s^2-4s-5\right)} = \frac{As+B}{\left(s^2+16\right)} + \frac{C}{(s+1)} + \frac{D}{(s-5)}$$

$$\frac{s}{\left(s^2+16\right)\left(s^2-4s-5\right)} = \frac{(As+B)(s+1)(s-5) + C\left(s^2+16\right)(s-5) + D\left(s^2+16\right)(s+1)}{\left(s^2+16\right)(s+1)(s-5)}$$

$$s = (As+B)(s+1)(s-5) + C\left(s^2+16\right)(s-5) + D\left(s^2+16\right)(s+1)$$

Nullstellen einsetzen:

$$s = 5 \quad \Rightarrow 5 = 246D$$

$$s = -1 \quad \Rightarrow -1 = -102C$$

Koeff.-Vergleich:

$$s^3 : 0 = A + C + D$$

$$s^0 : 0 = -5B - 80C + 16D$$

$$A = -\frac{21}{697}; \quad B = -\frac{64}{697}; \quad C = \frac{1}{102}; \quad D = \frac{5}{246}$$

$$X(s) = \frac{-\frac{21}{697}s - \frac{64}{697}}{\left(s^2+16\right)} + \frac{1}{102}\frac{1}{(s+1)} + \frac{5}{246}\frac{1}{(s-5)}$$

$$X(s) = -\frac{21}{697}\frac{s}{\left(s^2+16\right)} - \frac{16}{697}\frac{4}{\left(s^2+16\right)} + \frac{1}{102}\frac{1}{(s+1)} + \frac{5}{246}\frac{1}{(s-5)} \qquad \Big| \mathcal{L}^{-1}\{\ldots\}$$

$$x(t) = -\frac{21}{697}\cos(4t) - \frac{16}{697}\sin(4t) + \frac{1}{102}e^{-t} + \frac{5}{246}e^{5t}$$

2.5.6. $\ddot{x} - 4\dot{x} - 5x = 7$; $\quad x(0) = 0, \dot{x}(0) = 0$

$$\ddot{x} - 4\dot{x} - 5x = 7 \quad \Big| L\{\ldots\}$$

$$\frac{7}{s} = s^2X(s) - sx(0) - \dot{x}(0) - 4sX(s) + 4x(0) - 5X(s)$$

$$\frac{7}{s} = s^2X(s) - 4sX(s) - 5X(s)$$

$$\frac{7}{s} = X(s)\left(s^2 - 4s - 5\right)$$

$$X(s) = \frac{7}{s\left(s^2 - 4s - 5\right)} = \frac{7}{s(s+1)(s-5)}$$

$$X(s) = \frac{A}{s+1} + \frac{B}{s-5} + \frac{C}{s}$$

$$X(s) = \frac{As(s-5) + Bs(s+1) + C(s+1)(s-5)}{s(s+1)(s-5)}$$

Zählervergleich :

$$7 = As(s-5) + Bs(s+1) + C(s+1)(s-5)$$

Nullstellen einsetzen:

$$s = 5 \quad \Rightarrow 7 = 30B$$

$$s = 0 \quad \Rightarrow 7 = -5C$$

$$s = -1 \quad \Rightarrow 7 = 6A$$

$$A = \frac{7}{6}; \quad B = \frac{7}{30}; \quad C = -\frac{7}{5}$$

$$X(s) = \frac{7}{6}\frac{1}{s+1} + \frac{7}{30}\frac{1}{s-5} - \frac{7}{5}\frac{1}{s} \quad \Big| L^{-1}\{\ldots\}$$

$$x(t) = \frac{7}{6}e^{-t} + \frac{7}{30}e^{5t} - \frac{7}{5}$$

2.5.7. $\ddot{x} - 4\dot{x} - 5x = t^2$; $x(0) = 0, \dot{x}(0) = 0$

$$\ddot{x} - 4\dot{x} - 5x = t^2 \quad \left| L\{\ldots\}\right.$$

$$\frac{2}{s^3} = s^2X(s) - sx(0) - \dot{x}(0) - 4sX(s) + 4x(0) - 5X(s)$$

$$\frac{2}{s^3} = s^2X(s) - 4sX(s) - 5X(s)$$

$$\frac{2}{s^3} = X(s)\left(s^2 - 4s - 5\right)$$

$$X(s) = \frac{2}{s^3\left(s^2 - 4s - 5\right)}$$

$$\frac{2}{s^3\left(s^2 - 4s - 5\right)} = \frac{A}{s+1} + \frac{B}{s-5} + \frac{C}{s} + \frac{D}{s^2} + \frac{E}{s^3}$$

$$\frac{2}{s^3\left(s^2 - 4s - 5\right)} = \frac{As^3(s-5) + Bs^3(s+1) + Cs^2\left(s^2 - 4s - 5\right) + Ds\left(s^2 - 4s - 5\right) + E\left(s^2 - 4s - 5\right)}{s^3\left(s^2 - 4s - 5\right)}$$

$$2 = A\left(s^4 - 5s^3\right) + B\left(s^4 + s^3\right) + C\left(s^4 - 4s^3 - 5s^2\right) + D\left(s^3 - 4s^2 - 5s\right) + E\left(s^2 - 4s - 5\right)$$

Nullstellen einsetzen:

$$s = 5 \quad \Rightarrow 2 = 750B$$

$$s = -1 \Rightarrow 2 = 6A$$

$$s = 0 \quad \Rightarrow 2 = -5E$$

Koeff.-Vergleich:

$$s^4 : 0 = A + B + C$$

$$s^1 : 0 = -5D - 4E$$

$$A = \frac{1}{3}; \quad B = \frac{1}{375}; \quad C = -\frac{42}{125}; \quad D = \frac{8}{25}; \quad E = -\frac{2}{5}$$

$$X(s) = \frac{1}{3}\frac{1}{s+1} + \frac{1}{375}\frac{1}{s-5} - \frac{42}{125}\frac{1}{s} + \frac{8}{25}\frac{1}{s^2} - \frac{2}{5}\frac{1}{s^3} \qquad \left| L^{-1}\{\ldots\}\right.$$

$$x(t) = \frac{1}{3}e^{-t} + \frac{1}{375}e^{5t} - \frac{42}{125} + \frac{8}{25}t - \frac{1}{5}t^2$$

2.5.8. $\ddot{x} - 4\dot{x} + 40x = \sin(t)$; $x(0) = 0,\ \dot{x}(0) = 0$

$$\ddot{x} - 4\dot{x} + 40x = \sin(t) \quad \Big| L\{\dots\}$$

$$\frac{1}{s^2+1} = s^2X(s) - sx(0) - \dot{x}(0) - 4sX(s) + 4x(0) + 40X(s)$$

$$\frac{1}{s^2+1} = s^2X(s) - 4sX(s) + 40X(s)$$

$$\frac{1}{s^2+1} = X(s)\left(s^2 - 4s + 40\right)$$

$$X(s) = \frac{1}{\left(s^2+1\right)\left(s^2-4s+40\right)}$$

$$s^2 - 4s + 40 = 0 \qquad \Rightarrow \qquad s_{1/2} = 2 \pm 6j$$

$$\frac{1}{\left(s^2+1\right)\left(s^2-4s+40\right)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-4s+40}$$

$$\frac{1}{\left(s^2+1\right)\left(s^2-4s+40\right)} = \frac{(As+B)\left(s^2-4s+40\right) + (Cs+D)\left(s^2+1\right)}{\left(s^2+1\right)\left(s^2-4s+40\right)}$$

$$1 = A\left(s^3 - 4s^2 + 40s\right) + B\left(s^2 - 4s + 40\right) + C\left(s^3 + s\right) + D\left(s^2 + 1\right)$$

Koeff.-Vergleich:

$$s^3: 0 = A + C$$

$$s^2: 0 = -4A + B + D$$

$$s^1: 0 = 40A - 4B + C$$

$$s^0: 1 = 40B + D$$

$$A = \frac{4}{1537}; \quad B = \frac{39}{1537}; \quad C = -\frac{4}{1537}; \quad D = -\frac{23}{1537}$$

$$X(s) = \frac{\frac{4}{1537}s + \frac{39}{1537}}{s^2+1} + \frac{-\frac{4}{1537}s - \frac{23}{1537}}{s^2-4s+40}$$

$$X(s) = \frac{4}{1537}\frac{s}{s^2+1} + \frac{39}{1537}\frac{1}{s^2+1} - \frac{\frac{4}{1537}(s-2) + 2\frac{4}{1537} + \frac{23}{1537}}{(s-2)^2+36}$$

$$X(s) = \frac{4}{1537}\frac{s}{s^2+1} + \frac{39}{1537}\frac{1}{s^2+1} - \frac{4}{1537}\frac{(s-2)}{(s-2)^2+36} - \frac{31}{1537}\cdot\frac{1}{6}\cdot\frac{6}{(s-2)^2+36} \quad \Big| L^{-1}\{\dots\}$$

$$x(t) = \frac{1}{1537}\left(4\cos(t) + 39\sin(t)\right) - \frac{e^{2t}}{1537}\left(4\cos(6t) + \frac{31}{6}\sin(6t)\right)$$

2.5.9. $\dot{y} + 3y = 2t + 4e^{-t}$; $y(0) = 0,\ \dot{y}(0) = 0$

$$\dot{y} + 3y = 2t + 4e^{-t} \quad \Big| L\{\ldots\}$$

$$2\frac{1}{s^2} + 4\frac{1}{s+1} = sY(s) - y(0) + 3Y(s)$$

$$2\frac{1}{s^2} + 4\frac{1}{s+1} = sY(s) + 3Y(s)$$

$$\frac{2(s+1) + 4s^2}{s^2(s+1)} = Y(s)(s+3)$$

$$Y(s) = \frac{2(s+1) + 4s^2}{s^2(s+1)(s+3)} = \frac{4s^2 + 2s + 2}{s^2(s+1)(s+3)}$$

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s} + \frac{D}{s^2}$$

$$Y(s) = \frac{As^2(s+3) + Bs^2(s+1) + Cs(s+1)(s+3) + D(s+1)(s+3)}{s^2(s+1)(s+3)}$$

$$4s^2 + 2s + 2 = As^2(s+3) + Bs^2(s+1) + Cs(s+1)(s+3) + D(s+1)(s+3)$$

Nullstellen einsetzen:

$$s = -3 \;\Rightarrow 32 = -18B$$

$$s = -1 \;\Rightarrow 4 = 2A$$

$$s = 0 \quad \Rightarrow 2 = 3D$$

Koeff.-Vergleich:

$$s^3 : 0 = A + B + C$$

$$A = 2; \quad B = -\frac{16}{9}; \quad C = -\frac{2}{9}; \quad D = \frac{2}{3}$$

$$Y(s) = 2\frac{1}{s+1} - \frac{16}{9}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{s} + \frac{2}{3}\cdot\frac{1}{s^2} \quad \Big| L^{-1}\{\ldots\}$$

$$y(t) = 2e^{-t} - \frac{16}{9}e^{-3t} - \frac{2}{9} + \frac{1}{3}t$$

Achtung !!!

Die zweite Anfangsbedingung $\dot{y}(0) = 0$ wird nicht erfüllt. Überzeugen Sie sich davon, indem Sie die erste Ableitung von y bilden und t=0 einsetzen.

Für die spezielle Lösung einer DGL 1. Ordnung ist i.a. nur eine Zusatzbedingung notwendig. Weitere Bedingungen könnten dann nur rein zufällig auch erfüllt sein, was aber hier nicht zutrifft.

2.5.10. $\dot{y} + 3y = \sin(2t)$; $\quad y(0) = 1,\ \dot{y}(0) = 0$

$$\dot{y} + 3y = \sin(2t) \quad \Big| L\{\ldots\}$$
$$\frac{2}{s^2+4} = sY(s) - y(0) + 3Y(s)$$
$$\frac{2}{s^2+4} = sY(s) - 1 + 3Y(s) \quad \Big| +1$$
$$\frac{s^2+6}{s^2+4} = Y(s)(s+3)$$
$$Y(s) = \frac{s^2+6}{(s^2+4)(s+3)}$$
$$Y(s) = \frac{As+B}{s^2+4} + \frac{C}{s+3}$$
$$Y(s) = \frac{(As+B)(s+3) + C(s^2+4)}{(s^2+4)(s+3)}$$

Zählervergleich:
$$s^2 + 6 = (As+B)(s+3) + C(s^2+4)$$

Nullstelle einsetzen:
$$s = -3 \;\Rightarrow 15 = 13C$$

Koeff.-Vergleich:
$$s^2: \quad 1 = A + C$$
$$s^0: \quad 6 = 3B + 4C$$

$$A = -\frac{2}{13}; \quad B = \frac{6}{13}; \quad C = \frac{15}{13}$$

$$Y(s) = \frac{-\frac{2}{13}s + \frac{6}{13}}{s^2+4} + \frac{15}{13}\frac{1}{s+3}$$
$$Y(s) = -\frac{2}{13}\frac{s}{s^2+4} + \frac{6}{13}\cdot\frac{1}{2}\frac{2}{s^2+4} + \frac{15}{13}\frac{1}{s+3} \Big| L^{-1}\{\ldots\}$$

$$y(t) = -\frac{2}{13}\cos(2t) + \frac{3}{13}\sin(2t) + \frac{15}{13}e^{-3t}$$

Achtung !!! Wie in 2.5.9 ist die Anfangsbedingung $\dot{y}(0) = 0$ nicht erfüllt !

2.5.11*. $\ddot{y}+\ddot{y}+\dot{y}-14y=t$; $y(0)=\dot{y}(0)=\ddot{y}(0)=0$

$$t=\ddot{y}+\ddot{y}+\dot{y}-14y \quad \Big| L\{\ldots\}$$

$$\frac{1}{s^2}=s^3Y(s)-s^2y(0)-s\dot{y}(0)-\ddot{y}(0)+s^2Y(s)-sy(0)-\dot{y}(0)$$
$$+sY(s)-\dot{y}(0)-14Y(s)$$

$$\frac{1}{s^2}=s^3Y(s)+s^2Y(s)+sY(s)-14Y(s)=Y(s)\left(s^3+s^2+s-14\right)$$

$$Y(s)=\frac{1}{s^2\left(s^3+s^2+s-14\right)} \qquad \text{Nenner}-\text{Nullstelle } s=2$$

$$Y(s)=\frac{1}{s^2\left(s-2\right)\left(s^2+3s+7\right)} \qquad \text{konj.komplexe Nullstellen}$$

$$Y(s)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-2}+\frac{Ds+E}{s^2+3s+7}$$

$$Y(s)=\frac{As(s-2)\left(s^2+3s+7\right)+B(s-2)\left(s^2+3s+7\right)+Cs^2\left(s^2+3s+7\right)+s^2(Ds+E)(s-2)}{s^2(s-2)\left(s^2+3s+7\right)}$$

Zählervergleich:

$$1=As(s-2)\left(s^2+3s+7\right)+B(s-2)\left(s^2+3s+7\right)+Cs^2\left(s^2+3s+7\right)+s^2(Ds+E)(s-2)$$

Nullstellen: $s=0 \quad \Rightarrow 1=-14B$ $\qquad s=2 \quad \Rightarrow 1=68C$

K.-Vergleich: s^4: $0=A+C+D$

: s^3: $0=A+B+3C-2D+E$

: s^1: $0=-14A+B$

$$A=-\frac{1}{196};\quad B=-\frac{1}{14};\quad C=\frac{1}{68};\quad D=-\frac{8}{833};\quad E=\frac{11}{833}$$

$$Y(s)=-\frac{1}{196}\frac{1}{s}-\frac{1}{14}\frac{1}{s^2}+\frac{1}{68}\frac{1}{s-2}+\frac{-\frac{8}{833}s+\frac{11}{833}}{(s+1,5)^2+\frac{19}{4}}$$

$$Y(s)=-\frac{1}{196}\frac{1}{s}-\frac{1}{14}\frac{1}{s^2}+\frac{1}{68}\frac{1}{s-2}+\frac{-\frac{8}{833}(s+1,5)+\frac{1,5\cdot 8}{833}+\frac{11}{833}}{(s+1,5)^2+\frac{19}{4}}$$

$$Y(s)=-\frac{1}{196}\frac{1}{s}-\frac{1}{14}\frac{1}{s^2}+\frac{1}{68}\frac{1}{s-2}-\frac{8}{833}\frac{(s+1,5)}{(s+1,5)^2+\frac{19}{4}}+\frac{23}{833}\cdot\frac{2}{\sqrt{19}}\frac{\sqrt{19}/2}{(s+1,5)^2+\frac{19}{4}}$$

$$y(t)=-\frac{1}{196}-\frac{1}{14}t+\frac{1}{68}e^{2t}-\frac{e^{1,5t}}{833}\left[8\cos\left(\frac{\sqrt{19}}{2}t\right)-\frac{46}{\sqrt{19}}\sin\left(\frac{\sqrt{19}}{2}t\right)\right]$$

Aufgabe 2.6 Transformieren Sie in den Zeitbereich unter Verwendung der Faltungsregel!

2.6.1. $F(s) = \frac{1}{s^3 + 4s}$

$$F(s) = \frac{1}{s^3 + 4s} = \frac{1}{s}\frac{1}{s^2 + 4}$$

$$F_1(s) = \frac{1}{s} \qquad f_1(t) = 1 \qquad f_1(\xi) = 1$$

$$F_2(s) = \frac{1}{s^2 + 4} = \frac{1}{2} \cdot \frac{2}{s^2 + 2^2} \qquad f_2(t) = \frac{1}{2}\sin(2t) \qquad f_2(t-\xi) = \frac{1}{2}\sin(2(t-\xi))$$

$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(\xi) f_2(t-\xi)\, d\xi$$

$$f(t) = \int_0^t \left(1 \cdot \frac{1}{2}\sin(2(t-\xi))\right) d\xi$$

$$f(t) = \frac{1}{2}\int_0^t \sin(-2\xi + 2t)\, d\xi$$

$$f(t) = \frac{1}{2}\left[\frac{1}{2}\cos(-2\xi + 2t)\right]_0^t$$

$$f(t) = \frac{1}{4}(\cos(0) - \cos(2t))$$

$$f(t) = \frac{1}{4} - \frac{1}{4}\cos(2t)$$

2.6.2. $$F(s) = \frac{1}{(s+2)(s+3)^2}$$

$$F(s) = \frac{1}{(s+2)} \cdot \frac{1}{(s+3)^2}$$

$F_1(s) = \frac{1}{(s+2)}$ $\quad f_1(t) = e^{-2t}$ $\quad f_1(t-\xi) = e^{-2(t-\xi)}$

$F_2(s) = \frac{1}{(s+3)^2}$ $\quad f_2(t) = t \cdot e^{-3t}$ $\quad f_2(\xi) = \xi \cdot e^{-3\xi}$

$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\xi) f_2(\xi) d\xi$ $\quad f_1, f_2$ kommutativ !!

$$f(t) = \int_0^t \left(e^{-2(t-\xi)} \xi e^{-3\xi}\right) d\xi$$

$$f(t) = \int_0^t \left(\xi e^{-2t} e^{2\xi} e^{-3\xi}\right) d\xi$$

$$f(t) = e^{-2t} \int_0^t \left(\xi e^{-\xi}\right) d\xi$$

$$f(t) = e^{-2t} \left[-(\xi+1) e^{-\xi}\right]_0^t$$

$$f(t) = e^{-2t} \left(-(t+1) e^{-t} + 1\right)$$

$$f(t) = e^{-2t} - te^{-3t} - e^{-3t}$$